55. Multiple Choice Let $y = uv$ be the product of the functions u and v. Find $y'(1)$ if $u(1) = 2$, $u'(1) = 3$, $v(1) = -1$, and $v'(1) = 1$.

(A) -4 (B) -1 (C) 1 (D) 4 (E) 7

56. Multiple Choice Let $f(x) = x - \frac{1}{x}$. Find $f''(x)$.

(A) $1 + \frac{1}{x^2}$ (B) $1 - \frac{1}{x^2}$ (C) $\frac{2}{x^3}$

(D) $-\frac{2}{x^3}$ (E) does not exist

57. Multiple Choice Which of the following is $\frac{d}{dx}\left(\frac{x+1}{x-1}\right)$?

(A) $\frac{2}{(x-1)^2}$ (B) 0 (C) $-\frac{x^2+1}{x^2}$

(D) $2x - \frac{1}{x^2} - 1$ (E) $-\frac{2}{(x-1)^2}$

58. Multiple Choice Assume $f(x) = (x^2 - 1)(x^2 + 1)$. Which of the following gives the number of horizontal tangents of f?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Extending the Ideas

59. *Leibniz's Proof of the Product Rule* Here's how Leibniz explained the Product Rule in a letter to his colleague John Wallis:

It is useful to consider quantities infinitely small such that when their ratio is sought, they may not be considered zero, but which are rejected as often as they occur with quantities incomparably greater. Thus if we have $x + dx$, dx is rejected. Similarly we cannot have xdx and $dxdx$ standing together, as xdx is incomparably greater than $dxdx$. Hence if we are to differentiate uv, we write

$$\begin{aligned} d(uv) &= (u + du)(v + dv) - uv \\ &= uv + vdu + udv + dudv - uv \\ &= vdu + udv. \end{aligned}$$

Answer the following questions about Leibniz's proof.

(a) What does Leibniz mean by a quantity being "rejected"?

(b) What happened to $dudv$ in the last step of Leibniz's proof?

(c) Divide both sides of Leibniz's formula

$$d(uv) = vdu + udv$$

by the differential dx. What formula results?

(d) Why would the critics of Leibniz's time have objected to dividing both sides of the equation by dx?

(e) Leibniz had a similar simple (but not-so-clean) proof of the Quotient Rule. Can you reconstruct it?

Quick Quiz for AP* Preparation: Sections 3.1–3.3

1. Multiple Choice Let $f(x) = |x + 1|$. Which of the following statements about f are true?

I. f is continuous at $x = -1$.

II. f is differentiable at $x = -1$.

III. f has a corner at $x = -1$.

(A) I only (B) II only (C) III only

(D) I and III only (E) I and II only

2. Multiple Choice If the line normal to the graph of f at the point $(1, 2)$ passes through the point $(-1, 1)$, then which of the following gives the value of $f'(1)$ = ?

(A) -2 (B) 2 (C) $-1/2$ (D) $1/2$ (E) 3

3. Multiple Choice Find dy/dx if $y = \frac{4x-3}{2x+1}$.

(A) $\frac{10}{(4x-3)^2}$ (B) $-\frac{10}{(4x-3)^2}$ (C) $\frac{10}{(2x+1)^2}$

(D) $-\frac{10}{(2x+1)^2}$ (E) 2

4. Free Response Let $f(x) = x^4 - 4x^2$.

(a) Find all the points where f has horizontal tangents.

(b) Find an equation of the tangent line at $x = 1$.

(c) Find an equation of the normal line at $x = 1$.

3.4 Velocity and Other Rates of Change

What you will learn about . . .

- Instantaneous Rates of Change
- Motion Along a Line
- Sensitivity to Change
- Derivatives in Economics

and why . . .

Derivatives give the rates at which things change in the world.

Instantaneous Rates of Change

In this section we examine some applications in which derivatives as functions are used to represent the rates at which things change in the world around us. It is natural to think of change as change with respect to time, but other variables can be treated in the same way. For example, a physician may want to know how change in dosage affects the body's response to a drug. An economist may want to study how the cost of producing steel varies with the number of tons produced.

If we interpret the difference quotient

$$\frac{f(x+h) - f(x)}{h}$$

as the average rate of change of the function f over the interval from x to $x + h$, we can interpret its limit as h approaches 0 to be the rate at which f is changing at the point x.

DEFINITION Instantaneous Rate of Change

The **(instantaneous) rate of change** of f with respect to x at a is the derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h},$$

provided the limit exists.

It is conventional to use the word *instantaneous* even when x does not represent time. The word, however, is frequently omitted in practice. When we say *rate of change*, we mean *instantaneous rate of change*.

EXAMPLE 1 Enlarging Circles

(a) Find the rate of change of the area A of a circle with respect to its radius r.

(b) Evaluate the rate of change of A at $r = 5$ and at $r = 10$.

(c) If r is measured in inches and A is measured in square inches, what units would be appropriate for dA/dr?

SOLUTION

The area of a circle is related to its radius by the equation $A = \pi r^2$.

(a) The (instantaneous) rate of change of A with respect to r is

$$\frac{dA}{dr} = \frac{d}{dr}(\pi r^2) = \pi \cdot 2r = 2\pi r.$$

(b) At $r = 5$, the rate is 10π (about 31.4). At $r = 10$, the rate is 20π (about 62.8).

Notice that the rate of change gets bigger as r gets bigger. As can be seen in Figure 3.21, the same change in radius brings about a bigger change in area as the circles grow radially away from the center.

(c) The appropriate units for dA/dr are square inches (of area) per inch (of radius).

Now Try Exercise 1.

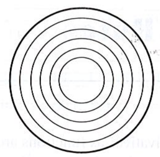


Figure 3.21 The same change in radius brings about a larger change in area as the circles grow radially away from the center. (Example 1, Exploration 1)

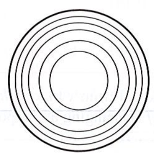


Figure 3.22 Which is the more appropriate model for the growth of rings in a tree—the circles here or those in Figure 3.21? (Exploration 1)

EXPLORATION 1 Growth Rings on a Tree

The phenomenon observed in Example 1, that the rate of change in area of a circle with respect to its radius gets larger as the radius gets larger, is reflected in nature in many ways. When trees grow, they add layers of wood directly under the inner bark during the growing season, then form a darker, protective layer for protection during the winter. This results in concentric rings that can be seen in a cross-sectional slice of the trunk. The age of the tree can be determined by counting the rings.

1. Look at the concentric rings in Figure 3.21 and Figure 3.22. Which is a better model for the pattern of growth rings in a tree? Is it likely that a tree could find the nutrients and light necessary to increase its amount of growth every year?
2. Considering how trees grow, explain why the change in *area* of the rings remains relatively constant from year to year.
3. If the change in area is constant, and if

$$\frac{dA}{dr} = \frac{\text{change in area}}{\text{change in radius}} = 2\pi r,$$

explain why the change in radius must get smaller as r gets bigger.

Motion Along a Line

Suppose that an object is moving along a coordinate line (say an s-axis) so that we know its position s on that line as a function of time t:

$$s = f(t).$$

The **displacement** of the object over the time interval from t to $t + \Delta t$ is

$$\Delta s = f(t + \Delta t) - f(t)$$

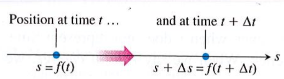


Figure 3.23 The positions of an object moving along a coordinate line at time t and shortly later at time $t + \Delta t$.

(Figure 3.23) and the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the object's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt shrinks to zero. The limit is the derivative of f with respect to t.

DEFINITION Instantaneous Velocity

The **(instantaneous) velocity** is the derivative of the position function $s = f(t)$ with respect to time. At time t the velocity is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

EXAMPLE 2 Finding the Velocity of a Race Car

Figure 3.24 shows the time-to-distance graph of a 1996 Riley & Scott Mk III-Olds WSC race car. The slope of the secant PQ is the average velocity for the 3-second interval from $t = 2$ to $t = 5$ sec, in this case, about 100 ft/sec or 68 mph. The slope of the tangent at P is the speedometer reading at $t = 2$ sec, about 57 ft/sec or 39 mph. The acceleration for the period shown is a nearly constant 28.5 ft/sec during each second, which is about $0.89g$ where g is the acceleration due to gravity. The race car's top speed is an estimated 190 mph. *Source: Road and Track,* March 1997.

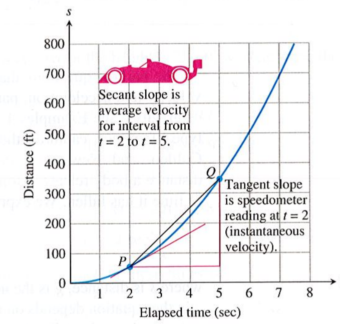


Figure 3.24 The time-to-distance graph for Example 2.

Now Try Exercise 7.

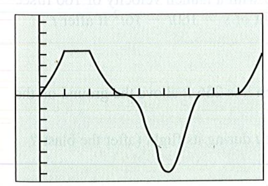
Besides telling how fast an object is moving, velocity tells the direction of motion. When the object is moving forward (when s is increasing), the velocity is positive; when the object is moving backward (when s is decreasing), the velocity is negative.

If we drive to a friend's house and back at 30 mph, the speedometer will show 30 on the way over but will not show -30 on the way back, even though our distance from home is decreasing. The speedometer always shows *speed,* which is the absolute value of velocity. Speed measures the rate of motion regardless of direction.

DEFINITION Speed

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$



[–4, 36] by [–7.5, 7.5]

Figure 3.25 A student's velocity graph from data recorded by a motion detector. (Example 3)

EXAMPLE 3 Reading a Velocity Graph

A student walks around in front of a motion detector that records her velocity at 1-second intervals for 36 seconds. She stores the data in her graphing calculator and uses it to generate the time-velocity graph shown in Figure 3.25. Describe her motion as a function of time by reading the velocity graph. When is her *speed* a maximum?

SOLUTION

The student moves forward for the first 14 seconds, moves backward for the next 12 seconds, stands still for 6 seconds, and then moves forward again. She achieves her maximum speed at $t \approx 20$, while moving backward.

Now Try Exercise 9.

The rate at which a body's velocity changes is called the body's *acceleration*. The acceleration measures how quickly the body picks up or loses speed.

DEFINITION Acceleration

Acceleration is the derivative of velocity with respect to time. If a body's velocity at time t is $v(t) = ds/dt$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

The earliest questions that motivated the discovery of calculus were concerned with velocity and acceleration, particularly the motion of freely falling bodies under the force of gravity. (See Examples 1 and 2 in Section 2.1.) The mathematical description of this type of motion captured the imagination of many great scientists, including Aristotle, Galileo, and Newton. Experimental and theoretical investigations revealed that the distance a body released from rest falls freely is proportional to the square of the amount of time it has fallen. We express this by saying that

$$s = \frac{1}{2}gt^2,$$

where s is distance, g is the acceleration due to Earth's gravity, and t is time. The value of g in the equation depends on the units used to measure s and t. With t in seconds (the usual unit), we have the following values:

Free-fall Constants (Earth)

English units: $g = 32\dfrac{\text{ft}}{\text{sec}^2}$, $s = \dfrac{1}{2}(32)t^2 = 16t^2$ (s in feet)

Metric units: $g = 9.8\dfrac{\text{m}}{\text{sec}^2}$, $s = \dfrac{1}{2}(9.8)t^2 = 4.9t^2$ (s in meters)

The abbreviation ft/sec^2 is read "feet per second squared" or "feet per second per second," and m/sec^2 is read "meters per second squared."

EXAMPLE 4 Modeling Vertical Motion

A dynamite blast propels a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Figure 3.26a). It reaches a height of $s = 160t - 16t^2$ ft after t seconds.

(a) How high does the rock go?

(b) What is the velocity and speed of the rock when it is 256 ft above the ground on the way up? on the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground?

SOLUTION

In the coordinate system we have chosen, s measures height from the ground up, so velocity is positive on the way up and negative on the way down.

continued

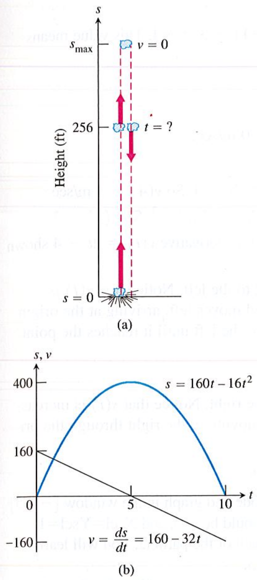


Figure 3.26 (a) The rock in Example 4. (b) The graphs of s and v as functions of time t, showing that s is largest when $v = ds/dt = 0$. (The graph of s is *not* the path of the rock; it is a plot of height as a function of time.) (Example 4)

(a) The instant when the rock is at its highest point is the one instant during the flight when the velocity is 0. At any time t, the velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec.}$$

The velocity is zero when $160 - 32t = 0$, or at $t = 5$ sec.

The maximum height is the height of the rock at $t = 5$ sec. That is,

$$s_{max} = s(5) = 160(5) - 16(5)^2 = 400 \text{ ft.}$$

See Figure 3.26b.

(b) To find the velocity when the height is 256 ft, we determine the two values of t for which $s(t) = 256$ ft.

$$\begin{aligned} s(t) = 160t - 16t^2 &= 256 \\ 16t^2 - 160t + 256 &= 0 \\ 16(t^2 - 10t + 16) &= 0 \\ (t-2)(t-8) &= 0 \\ t = 2 \text{ sec} \quad \text{or} \quad t &= 8 \text{ sec} \end{aligned}$$

The velocity of the rock at each of these times is

$$v(2) = 160 - 32(2) = 96 \text{ ft/sec},$$
$$v(8) = 160 - 32(8) = -96 \text{ ft/sec}.$$

At both instants, the speed of the rock is 96 ft/sec.

(c) At any time during its flight after the explosion, the rock's acceleration is

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

The acceleration is always downward. When the rock is rising, it is slowing down; when it is falling, it is speeding up.

(d) The rock hits the ground at the positive time for which $s = 0$. The equation $160t - 16t^2 = 0$ has two solutions: $t = 0$ and $t = 10$. The blast initiated the flight of the rock from ground level at $t = 0$. The rock returned to the ground 10 seconds later.

Now Try Exercise 13.

EXAMPLE 5 Studying Particle Motion

A particle moves along a line so that its position at any time $t \geq 0$ is given by the function $s(t) = t^2 - 4t + 3$, where s is measured in meters and t is measured in seconds.

$f(t+\Delta t) - f(t)$ f

(a) Find the displacement of the particle during the first 2 seconds.

(b) Find the average velocity of the particle during the first 4 seconds.

(c) Find the instantaneous velocity of the particle when $t = 4$.

(d) Find the acceleration of the particle when $t = 4$.

(e) Describe the motion of the particle. At what values of t does the particle change directions?

(f) Use parametric graphing to view the motion of the particle on the horizontal line $y = 2$.

continued

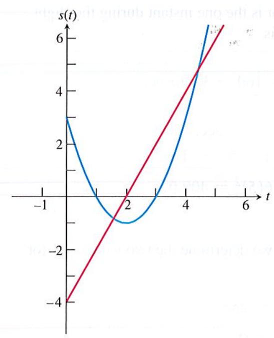
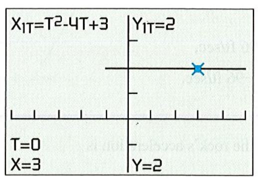


Figure 3.27 The graphs of $s(t) = t^2 - 4t + 3, t \geq 0$ (blue) and its derivative $v(t) = 2t - 4, t \geq 0$ (red). (Example 5)



$[-5, 5]$ by $[-2, 4]$

Figure 3.28 The graph of $X_{1T} = T^2 - 4T + 3$, $Y_{1T} = 2$ in parametric mode. (Example 5)

SOLUTION

(a) The displacement is given by $s(2) - s(0) = (-1) - 3 = -4$. This value means that the particle is 4 units left of where it started.

(b) The average velocity we seek is

$$\frac{s(4) - s(0)}{4 - 0} = \frac{3 - 3}{4} = 0 \text{ m/sec.}$$

(c) The velocity $v(t)$ at any time t is $v(t) = ds/dt = 2t - 4$. So $v(4) = 4$ m/sec

(d) The acceleration $a(t)$ at any time t is $a(t) = dv/dt = 2$ m/sec^2. So $a(4) = 2$.

(e) The graphs of $s(t) = t^2 - 4t + 3$ for $t \geq 0$ and its derivative $v(t) = 2t - 4$ shown in Figure 3.27 will help us analyze the motion.

For $0 \leq t < 2$, $v(t) < 0$, so the particle is moving to the left. Notice that $s(t)$ is decreasing. The particle starts ($t = 0$) at $s = 3$ and moves left, arriving at the origin $t = 1$ when $s = 0$. The particle continues moving to the left until it reaches the point $s = -1$ at $t = 2$.

At $t = 2$, $v = 0$, so the particle is at rest.

For $t > 2$, $v(t) > 0$, so the particle is moving to the right. Notice that $s(t)$ is increasing. In this interval, the particle starts at $s = -1$, moving to the right through the origin and continuing to the right for the rest of time.

The particle changes direction at $t = 2$ when $v = 0$.

(f) Enter $X_{1T}=T^2-4T+3$, $Y_{1T} = 2$ in parametric mode and graph in the window $[-5, 5]$ by $[-2, 4]$ with Tmin = 0, Tmax = 10 (it really should be ∞), and Xscl=Yscl=1 (Figure 3.28). By using TRACE you can follow the path of the particle. You will learn more ways to visualize motion in Explorations 2 and 3.

Now Try Exercise 19.

EXPLORATION 2 Modeling Horizontal Motion

The position (x-coordinate) of a particle moving on the horizontal line $y = 2$ is given by $x(t) = 4t^3 - 16t^2 + 15t$ for $t \geq 0$.

1. Graph the parametric equations $x_1(t) = 4t^3 - 16t^2 + 15t$, $y_1(t) = 2$ in $[-4, 6]$ by $[-3, 5]$. Use TRACE to support that the particle starts at the point $(0, 2)$, moves to the right, then to the left, and finally to the right. At what times does the particle reverse direction?
2. Graph the parametric equations $x_2(t) = x_1(t)$, $y_2(t) = t$ in the same viewing window. Explain how this graph shows the back and forth motion of the particle. Use this graph to find when the particle reverses direction.
3. Graph the parametric equations $x_3(t) = t$, $y_3(t) = x_1(t)$ in the same viewing window. Explain how this graph shows the back and forth motion of the particle. Use this graph to find when the particle reverses direction.
4. Use the methods in parts 1, 2, and 3 to represent and describe the *velocity* of the particle.

EXPLORATION 3 Seeing Motion on a Graphing Calculator

The graphs in Figure 3.26b give us plenty of information about the flight of the rock in Example 4, but neither graph shows the path of the rock in flight. We can simulate the moving rock by graphing the parametric equations

$$x_1(t) = 3(t < 5) + 3.1(t \geq 5), \quad y_1(t) = 160t - 16t^2$$

in dot mode.

This will show the upward flight of the rock along the vertical line $x = 3$, and the downward flight of the rock along the line $x = 3.1$.

1. To see the flight of the rock from beginning to end, what should we use for tMin and tMax in our graphing window?
2. Set xMin = 0, xMax = 6, and yMin = −10. Use the results from Example 4 to determine an appropriate value for yMax. (You will want the entire flight of the rock to fit within the vertical range of the screen.)
3. Set tStep initially at 0.1. (A higher number will make the simulation move faster. A lower number will slow it down.)
4. Can you explain why the grapher actually slows down when the rock would slow down, and speeds up when the rock would speed up?

Sensitivity to Change

When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is relatively **sensitive** to changes in x. The derivative $f'(x)$ is a measure of this sensitivity.

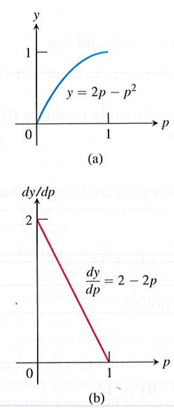


Figure 3.29 (a) The graph of $y = 2p - p^2$ describing the proportion of smooth-skinned peas. (b) The graph of dy/dp. (Example 6)

EXAMPLE 6 Sensitivity to Change

The Austrian monk Gregor Johann Mendel (1822–1884), working with garden peas and other plants, provided the first scientific explanation of hybridization. His careful records showed that if p (a number between 0 and 1) is the relative frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the relative frequency of the gene for wrinkled skin in peas (recessive), then the proportion of smooth-skinned peas in the next generation will be

$$y = 2p(1 - p) + p^2 = 2p - p^2.$$

Compare the graphs of y and dy/dp to determine what values of y are more sensitive to a change in p. The graph of y versus p in Figure 3.29a suggests that the value of y is more sensitive to a change in p when p is small than it is to a change in p when p is large. Indeed, this is borne out by the derivative graph in Figure 3.29b, which shows that dy/dp is close to 2 when p is near 0 and close to 0 when p is near 1.

Now Try Exercise 25.

Derivatives in Economics

Engineers use the terms *velocity* and *acceleration* to refer to the derivatives of functions describing motion. Economists, too, have a specialized vocabulary for rates of change and derivatives. They call them *marginals*.

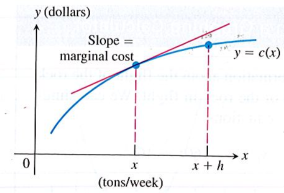


Figure 3.30 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons per week is $c(x + h) - c(x)$.

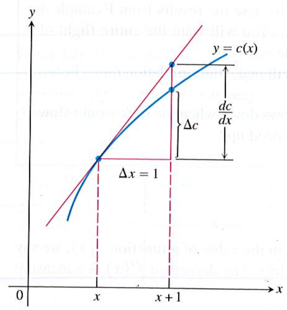


Figure 3.31 Because dc/dx is the slope of the tangent at x, the marginal cost dc/dx approximates the extra cost Δc of producing $\Delta x = 1$ more unit.

In a manufacturing operation, *the cost of production* $c(x)$ is a function of x, the number of units produced. The *marginal cost of production* is the rate of change of cost with respect to the level of production, so it is dc/dx.

Suppose $c(x)$ represents the dollars needed to produce x tons of steel in one week. It costs more to produce $x + h$ tons per week, and the cost difference divided by h is the average cost of producing each additional ton.

$$\frac{c(x+h) - c(x)}{h} = \left\{\begin{array}{l}\text{the average cost of each of the} \\ \text{additional } h \text{ tons produced}\end{array}\right.$$

The limit of this ratio as $h \to 0$ is the **marginal cost** of producing more steel per week when the current production is x tons (Figure 3.30).

$$\frac{dc}{dx} = \lim_{h \to 0} \frac{c(x+h) - c(x)}{h} = \text{marginal cost of production}$$

Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one more unit,

$$\frac{\Delta c}{\Delta x} = \frac{c(x+1) - c(x)}{1},$$

which is approximated by the value of dc/dx at x. This approximation is acceptable if the slope of c does not change quickly near x, for then the difference quotient is close to its limit dc/dx even if $\Delta x = 1$ (Figure 3.31). The approximation works best for large values of x.

EXAMPLE 7 Marginal Cost and Marginal Revenue

Suppose it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 10 radiators are produced, and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. Find the marginal cost and **marginal revenue.**

SOLUTION

The marginal cost of producing one more radiator a day when 10 are being produced is $c'(10)$.

$$c'(x) = \frac{d}{dx}(x^3 - 6x^2 + 15x) = 3x^2 - 12x + 15$$

$$c'(10) = 3(100) - 12(10) + 15 = 195 \text{ dollars}$$

The marginal revenue is

$$r'(x) = \frac{d}{dx}(x^3 - 3x^2 + 12x) = 3x^2 - 6x + 12,$$

so,

$$r'(10) = 3(100) - 6(10) + 12 = 252 \text{ dollars.}$$

Now Try Exercises 27 and 28.

Quick Review 3.4 *(For help, go to Sections 1.2, 3.1, and 3.3.)*

Exercise numbers with a gray background indicate problems that the authors have designed to be solved *without a calculator.*

In Exercises 1–10, answer the questions about the graph of the quadratic function $y = f(x) = -16x^2 + 160x - 256$ by analyzing the equation algebraically. Then support your answers graphically.

1. Does the graph open upward or downward?

2. What is the y-intercept?

3. What are the x-intercepts?

4. What is the range of the function?

5. What point is the vertex of the parabola?

6. At what x-values does $f(x) = 80$?

7. For what x-value does $dy/dx = 100$?

8. On what interval is $dy/dx > 0$?

9. Find $\displaystyle\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$.

10. Find d^2y/dx^2 at $x = 7$.

Section 3.4 Exercises

1. (a) Write the volume V of a cube as a function of the side length s.

(b) Find the (instantaneous) rate of change of the volume V with respect to a side s.

(c) Evaluate the rate of change of V at $s = 1$ and $s = 5$.

(d) If s is measured in inches and V is measured in cubic inches, what units would be appropriate for dV/ds?

2. (a) Write the area A of a circle as a function of the circumference C.

(b) Find the (instantaneous) rate of change of the area A with respect to the circumference C.

(c) Evaluate the rate of change of A at $C = \pi$ and $C = 6\pi$.

(d) If C is measured in inches and A is measured in square inches, what units would be appropriate for dA/dC?

3. (a) Write the area A of an equilateral triangle as a function of the side length s.

(b) Find the (instantaneous) rate of change of the area A with respect to a side s.

(c) Evaluate the rate of change of A at $s = 2$ and $s = 10$.

(d) If s is measured in inches and A is measured in square inches, what units would be appropriate for dA/ds?

4. A square of side length s is inscribed in a circle of radius r.

(a) Write the area A of the square as a function of the radius r of the circle.

(b) Find the (instantaneous) rate of change of the area A with respect to the radius r of the circle.

(c) Evaluate the rate of change of A at $r = 1$ and $r = 8$.

(d) If r is measured in inches and A is measured in square inches, what units would be appropriate for dA/dr?

Group Activity In Exercises 5 and 6, the coordinates s of a moving body for various values of t are given. **(a)** Plot s versus t on coordinate paper, and sketch a smooth curve through the given points. **(b)** Assuming that this smooth curve represents the motion of the body, estimate the velocity at $t = 1.0$, $t = 2.5$, and $t = 3.5$.

5.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
s (ft)	12.5	26	36.5	44	48.5	50	48.5	44	36.5

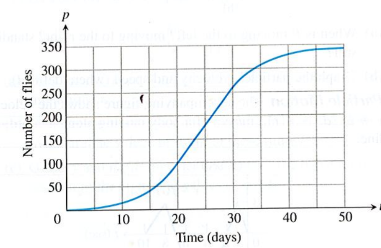
6.

t (sec)	0	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0
s (ft)	3.5	−4	−8.5	−10	−8.5	−4	3.5	14	27.5

7. Group Activity ***Fruit Flies*** *(Example 2, Section 2.4 continued)* Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

(a) Use the graphical technique of Section 3.1, Example 3, to graph the derivative of the fruit fly population introduced in Section 2.4. The graph of the population is reproduced below. What units should be used on the horizontal and vertical axes for the derivative's graph?

(b) During what days does the population seem to be increasing fastest? slowest?



8. *Draining a Tank* The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

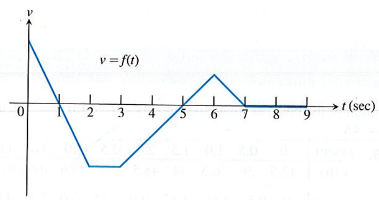
9. *Particle Motion* The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a coordinate line.

(a) When does the particle move forward? move backward? speed up? slow down?

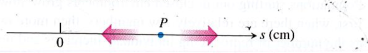
(b) When is the particle's acceleration positive? negative? zero?

(c) When does the particle move at its greatest speed?

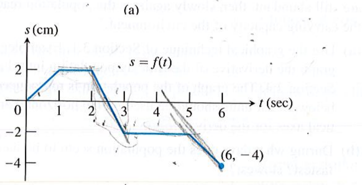
(d) When does the particle stand still for more than an instant?



10. *Particle Motion* A particle P moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of P as a function of time t.



(a)

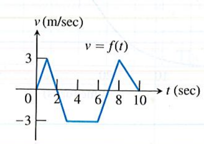


(b)

(a) When is P moving to the left? moving to the right? standing still?

(b) Graph the particle's velocity and speed (where defined).

11. *Particle Motion* The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/sec) of a body moving along a coordinate line.



(a) When does the body reverse direction?

(b) When (approximately) is the body moving at a constant speed?

(c) Graph the body's speed for $0 \le t \le 10$.

(d) Graph the acceleration, where defined.

12. *Thoroughbred Racing* A racehorse is running a 10-furlong race. (A furlong is 220 yards, although we will use furlongs and seconds as our units in this exercise.) As the horse passes each furlong marker (F), a steward records the time elapsed (t) since the beginning of the race, as shown in the table below:

F	0	1	2	3	4	5	6	7	8	9	10
t	0	20	33	46	59	73	86	100	112	124	135

(a) How long does it take the horse to finish the race?

(b) What is the average speed of the horse over the first 5 furlongs?

(c) What is the approximate speed of the horse as it passes the 3-furlong marker?

(d) During which portion of the race is the horse running the fastest?

(e) During which portion of the race is the horse accelerating the fastest?

13. *Lunar Projectile Motion* A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t seconds.

(a) Find the rock's velocity and acceleration as functions of time. (The acceleration in this case is the acceleration of gravity on the moon.)

(b) How long did it take the rock to reach its highest point?

(c) How high did the rock go?

(d) When did the rock reach half its maximum height?

(e) How long was the rock aloft?

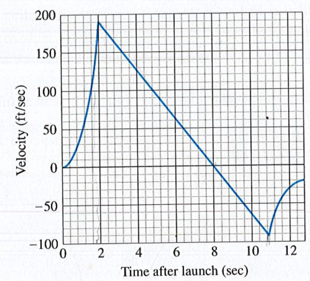
14. *Free Fall* The equations for free fall near the surfaces of Mars and Jupiter (s in meters, t in seconds) are: Mars, $s = 1.86t^2$; Jupiter, $s = 11.44t^2$. How long would it take a rock falling from rest to reach a velocity of 16.6 m/sec (about 60 km/h) on each planet?

15. *Projectile Motion* On Earth, in the absence of air, the rock in Exercise 13 would reach a height of $s = 24t - 4.9t^2$ meters in t seconds. How high would the rock go?

16. *Speeding Bullet* A bullet fired straight up from the moon's surface would reach a height of $s = 832t - 2.6t^2$ ft after t sec. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ ft after t sec. How long would it take the bullet to get back down in each case?

17. *Parametric Graphing* Devise a grapher simulation of the problem situation in Exercise 16. Use it to support the answers obtained analytically.

18. *Launching a Rocket* When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts downward. The parachute slows the rocket to keep it from breaking when it lands. This graph shows velocity data from the flight.



Use the graph to answer the following.

(a) How fast was the rocket climbing when the engine stopped?

(b) For how many seconds did the engine burn?

(c) When did the rocket reach its highest point? What was its velocity then?

(d) When did the parachute pop out? How fast was the rocket falling then?

(e) How long did the rocket fall before the parachute opened?

(f) When was the rocket's acceleration greatest? When was the acceleration constant?

19. *Particle Motion* A particle moves along a line so that its position at any time $t \ge 0$ is given by the function

$$s(t) = t^2 - 3t + 2,$$

where s is measured in meters and t is measured in seconds.

(a) Find the displacement during the first 5 seconds.

(b) Find the average velocity during the first 5 seconds.

(c) Find the instantaneous velocity when $t = 4$.

(d) Find the acceleration of the particle when $t = 4$.

(e) At what values of t does the particle change direction?

(f) Where is the particle when s is a minimum?

20. *Particle Motion* A particle moves along a line so that its position at any time $t \ge 0$ is given by the function $s(t) = -t^3 + 7t^2 - 14t + 8$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

21. *Particle Motion* A particle moves along a line so that its position at any time $t \ge 0$ is given by the function $s(t) = (t - 2)^2(t - 4)$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

22. *Particle Motion* A particle moves along a line so that its position at any time $t \ge 0$ is given by the function $s(t) = t^3 - 6t^2 + 8t + 2$ where s is measured in meters and t is measured in seconds.

(a) Find the instantaneous velocity at any time t.

(b) Find the acceleration of the particle at any time t.

(c) When is the particle at rest?

(d) Describe the motion of the particle. At what values of t does the particle change directions?

23. *Particle Motion* The position of a body at time t sec is $s = t^3 - 6t^2 + 9t$ m. Find the body's acceleration each time the velocity is zero.

24. *Finding Speed* A body's velocity at time t sec is $v = 2t^3 - 9t^2 + 12t - 5$ m/sec. Find the body's speed each time the acceleration is zero.

25. *Draining a Tank* It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

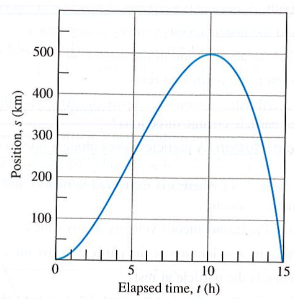
$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m.}$$

(a) Find the rate dy/dt (m/h) at which the water level is changing at time t.

(b) When is the fluid level in the tank falling fastest? slowest? What are the values of dy/dt at these times?

(c) Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt.

26. *Moving Truck* The graph here shows the position s of a truck traveling on a highway. The truck starts at $t = 0$ and returns 15 hours later at $t = 15$.



(a) Use the technique described in Section 3.1, Example 3, to graph the truck's velocity $v = ds/dt$ for $0 \le t \le 15$. Then repeat the process, with the velocity curve, to graph the truck's acceleration dv/dt.

(b) Suppose $s = 15t^2 - t^3$. Graph ds/dt and d^2s/dt^2, and compare your graphs with those in part (a).

27. *Marginal Cost* Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

(a) Find the average cost of producing 100 washing machines.

(b) Find the marginal cost when 100 machines are produced.

(c) Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

28. *Marginal Revenue* Suppose the weekly revenue in dollars from selling x custom-made office desks is

$$r(x) = 2000\left(1 - \frac{1}{x+1}\right).$$

(a) Draw the graph of r. What values of x make sense in this problem situation?

(b) Find the marginal revenue when x desks are sold.

(c) Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing sales from 5 desks a week to 6 desks a week.

(d) **Writing to Learn** Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

29. *Finding Profit* The monthly profit (in thousands of dollars) of a software company is given by

$$P(x) = \frac{10}{1 + 50 \cdot 2^{5-0.1x}},$$

where x is the number of software packages sold.

(a) Graph $P(x)$.

(b) What values of x make sense in the problem situation?

(c) Use NDER to graph $P'(x)$. For what values of x is P relatively sensitive to changes in x?

(d) What is the profit when the marginal profit is greatest?

(e) What is the marginal profit when 50 units are sold? 100 units, 125 units, 150 units, 175 units, and 300 units?

(f) What is $\lim_{x\to\infty} P(x)$? What is the maximum profit possible?

(g) **Writing to Learn** Is there a practical explanation to the maximum profit answer? Explain your reasoning.

30. In Step 1 of Exploration 2, at what time is the particle at the point $(5, 2)$?

31. Group Activity The graphs in Figure 3.31 show as functions of time t the position s, velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line. Which graph is which? Give reasons for your answers.

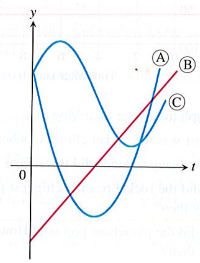


Figure 3.32 The graphs for Exercise 31.

32. Group Activity The graphs in Figure 3.33 show as functions of time t the position s, the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line. Which graph is which? Give reasons for your answers.

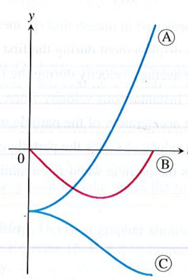


Figure 3.33 The graphs for Exercise 32.

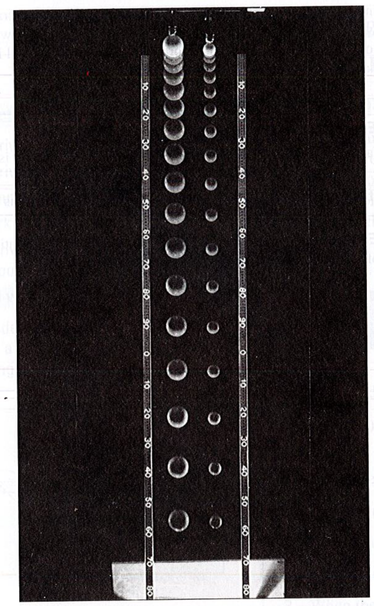


Figure 3.34 Two balls falling from rest. (Exercise 38)

33. *Pisa by Parachute* *(continuation of Exercise 18)* In 1988, Mike McCarthy parachuted 179 ft from the top of the Tower of Pisa. Make a rough sketch to show the shape of the graph of his downward velocity during the jump.

34. *Inflating a Balloon* The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

(a) At what rate does the volume change with respect to the radius when $r = 2$ ft?

(b) By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

35. *Volcanic Lava Fountains* Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 1900 ft straight into the air (a world record). What was the lava's exit velocity in feet per second? in miles per hour? [*Hint:* If v_0 is the exit velocity of a particle of lava, its height t seconds later will be $s = v_0t - 16t^2$ feet. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.]

36. Writing to Learn Suppose you are looking at a graph of velocity as a function of time. How can you estimate the acceleration at a given point in time?

37. *Particle Motion* The position (x-coordinate) of a particle moving on the line $y = 2$ is given by $x(t) = 2t^3 - 13t^2 + 22t - 5$ where t is time in seconds.

(a) Describe the motion of the particle for $t \ge 0$.

(b) When does the particle speed up? slow down?

(c) When does the particle change direction?

(d) When is the particle at rest?

(e) Describe the velocity and speed of the particle.

(f) When is the particle at the point $(5, 2)$?

38. *Falling Objects* The multiflash photograph in Figure 3.34 shows two balls falling from rest. The vertical rulers are marked in centimeters. Use the equation $s = 490t^2$ (the free-fall equation for s in centimeters and t in seconds) to answer the following questions.

(a) How long did it take the balls to fall the first 160 cm? What was their average velocity for the period?

(b) How fast were the balls falling when they reached the 160-cm mark? What was their acceleration then?

(c) About how fast was the light flashing (flashes per second)?

39. Writing to Learn Explain how the Sum and Difference Rule (Rule 4 in Section 3.3) can be used to derive a formula for *marginal profit* in terms of marginal revenue and marginal cost.

Standardized Test Questions

You may use a graphing calculator to solve the following problems.

40. True or False The speed of a particle at $t = a$ is given by the value of the velocity at $t = a$. Justify your answer.

41. True or False The acceleration of a particle is the second derivative of the position function. Justify your answer.

42. Multiple Choice Find the instantaneous rate of change of $f(x) = x^2 - 2/x + 4$ at $x = -1$.

(A) -7 (B) -4 (C) 0 (D) 4 (E) 7

43. Multiple Choice Find the instantaneous rate of change of the volume of a cube with respect to a side length x.

(A) x (B) $3x$ (C) $6x$ (D) $3x^2$ (E) x^3

In Exercises 44 and 45, a particle moves along a line so that its position at any time $t \ge 0$ is given by $s(t) = 2 + 7t - t^2$.

44. Multiple Choice At which of the following times is the particle moving to the left?

(A) $t = 0$ (B) $t = 1$ (C) $t = 2$ (D) $t = 7/2$ (E) $t = 4$

45. Multiple Choice When is the particle at rest?

(A) $t = 1$ (B) $t = 2$ (C) $t = 7/2$ (D) $t = 4$ (E) $t = 5$

Explorations

46. *Bacterium Population* When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while but then stopped growing and began to decline. The size of the population at time t (hours) was $b(t) = 10^6 + 10^4t - 10^3t^2$. Find the growth rates at $t = 0$, $t = 5$, and $t = 10$ hours.